A GENERALIZATION OF A TRACE INEQUALITY FOR POSITIVE DEFINITE MATRICES.

ELENA VERONICA BELMEGA, MARC JUNGERS, AND SAMSON LASAULCE

Abstract. In this note we generalize the trace inequality derived by [1] to the case where the number of terms of the sum (denoted by K) is arbitrary. More precisely we prove that $\mathcal{T}_K = \mathrm{Tr}\left\{\sum_{k=1}^{K}(\mathbf{A}_k - \mathbf{B}_k)\left[\left(\sum_{\ell=1}^{k}\mathbf{B}_\ell\right)^{-1} - \left(\sum_{\ell=1}^{k}\mathbf{A}_\ell\right)^{-1}\right]\right\} \geq 0$ for any set of positive definite matrices.

1. Introduction

Trace inequalities are useful in many areas like multiple input multiple output (MIMO) systems in control theory and communications. Proving the trace inequality under investigation in this note is a sufficient condition to ensure the uniqueness of a Nash equilibrium in certain MIMO communications game [2] where Rosen's diagonally strict concavity condition [3] is valid. The considered inequality has been proven by e.g., [4] for $K = 1$ and by [1] for $K = 2$. Here we generalize it to $K \in \mathbb{N}^*$. The main result of this note is as follows.

Theorem 1.1. *Let $K \in \mathbb{N}^*$. Assume that*

(i): $\mathbf{A}_1 = \mathbf{A}_1^H \succ 0$, $\mathbf{B}_1 = \mathbf{B}_1^H \succ 0$;

(ii): $\forall k \in \{2, \dots, K\}$, $\mathbf{A}_k = \mathbf{A}_k^H \succeq 0$ *and* $\mathbf{B}_k = \mathbf{B}_k^H \succeq 0$.

Then, we have that

$$\mathcal{T}_K \triangleq \mathrm{Tr}\left\{\sum_{k=1}^{K}(\mathbf{A}_k - \mathbf{B}_k)\left[\left(\sum_{\ell=1}^{k}\mathbf{B}_\ell\right)^{-1} - \left(\sum_{\ell=1}^{k}\mathbf{A}_\ell\right)^{-1}\right]\right\} \geq 0. \tag{1.1}$$

2. Auxiliary Results

In order to prove Theorem 1.1, we will use two auxiliary lemmas which are stated here for the sake of clarity.

Lemma 2.1. [1] *Let $\mathbf{A}$, $\mathbf{B}$ be two positive definite matrices, $\mathbf{C}$, $\mathbf{D}$, two positive semidefinite matrices whereas $\mathbf{X}$ is only assumed to be Hermitian. Then*

$$\mathrm{Tr}\left\{\mathbf{X}\mathbf{A}^{-1}\mathbf{X}\mathbf{B}^{-1}\right\} - \mathrm{Tr}\left\{\mathbf{X}(\mathbf{A}+\mathbf{C})^{-1}\mathbf{X}(\mathbf{B}+\mathbf{D})^{-1}\right\} \geq 0. \tag{2.1}$$

The proof is given in [1].

Lemma 2.2. *Let $\mathbf{A}$, $\mathbf{B}$ be two positive definite matrices, $\mathbf{C}$, $\mathbf{D}$, two positive semi-definite matrices. Then*

$$\begin{aligned} &\mathrm{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}\right\} \;=\\ &\mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\} \;\in\; \mathbb{R}. \end{aligned} \tag{2.2}$$

2000 *Mathematics Subject Classification.* 15A45.

Key words and phrases. Trace inequality, positive definite matrices, positive semidefinite matrices.

Proof. To prove the desired result, let us define $\mathcal{E}$ by

$$\mathcal{E} = \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})\left[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}\right]\right\} \tag{2.3}$$

and write it in two different ways.

$$\begin{array}{rcl} \mathcal{E} & = & \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}[\mathbf{A}+\mathbf{C}-\mathbf{B}-\mathbf{D}](\mathbf{A}+\mathbf{C})^{-1}\right\} \\ & = & \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}\right\}+ \\ & & \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\}. \end{array} \tag{2.4}$$

$$\begin{array}{rcl} \mathcal{E} & = & \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}[\mathbf{A}+\mathbf{C}-\mathbf{B}-\mathbf{D}](\mathbf{B}+\mathbf{D})^{-1}\right\} \\ & = & \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}\right\}+ \\ & & \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}\right\}. \end{array} \tag{2.5}$$

By using the commutation property of the trace and the two expressions of $\mathcal{E}$, we find the desired result. The only thing which needs to be proven is that $\mathcal{E}$ is real. For this purpose, observe that if we denote by $\mathbf{M} = (\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}$ then $\mathbf{M}^H = (\mathbf{A}+\mathbf{C})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})$ and from the result just proven we obtain that $\mathrm{Tr}(\mathbf{M}^H) = \mathrm{Tr}(\mathbf{M})$ and thus we have that $\mathrm{Tr}(\mathbf{M}) \in \mathbb{R}$. ☐

3. Proof of Theorem 1.1

Define $\mathbf{X}_k = \sum_{i=1}^{k} \mathbf{A}_i$, $\mathbf{Y}_k = \sum_{i=1}^{k} \mathbf{B}_i$, for all $k \geq 1$ which are both positive definite matrices. Notice that $\mathcal{T}_K$ can be re-written recursively as follows:

$$\begin{cases} \mathcal{T}_1 = \mathrm{Tr}\left\{(\mathbf{A}_1-\mathbf{B}_1)\mathbf{Y}_1^{-1}(\mathbf{A}_1-\mathbf{B}_1)\mathbf{X}_1^{-1}\right\} \\ \mathcal{T}_K = \mathcal{T}_{K-1} + \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\} + \\ \qquad \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\} \end{cases} \tag{3.1}$$

We proceed in two steps. First, we find a lower bound for $\mathcal{T}_K$ and then we prove that this bound is positive. First, let us prove that, for all $K \geq 1$:

$$\mathcal{T}_K \geq \frac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\} + \frac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{X}_K^{-1}\right\} \tag{3.2}$$

To this end we proceed by induction. For all $K \in \mathbb{N}^*$, define the proposition:

$$\mathcal{P}_K:\ \mathcal{T}_K \geq \frac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\} + \frac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{X}_K^{-1}\right\}. \tag{3.3}$$

It is easy to check that, for $K=1$, $\mathcal{P}_1$ is true:

$$\begin{array}{rcl} \mathcal{T}_1 & = & \mathrm{Tr}\left\{(\mathbf{A}_1-\mathbf{B}_1)\mathbf{Y}_1^{-1}(\mathbf{A}_1-\mathbf{B}_1)\mathbf{X}_1^{-1}\right\} \\ & = & \frac{1}{2}\mathrm{Tr}\left\{(\mathbf{A}_1-\mathbf{B}_1)\mathbf{Y}_1^{-1}(\mathbf{A}_1-\mathbf{B}_1)\mathbf{X}_1^{-1}\right\} + \frac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_1-\mathbf{Y}_1)\mathbf{Y}_1^{-1}(\mathbf{X}_1-\mathbf{Y}_1)\mathbf{X}_1^{-1}\right\}. \end{array} \tag{3.4}$$

Now, let us assume that $\mathcal{P}_{K-1}$ is true and then prove that this implies that $\mathcal{P}_K$ is also true. We have that:

$$
\begin{aligned}
\mathcal{T}_{K-1} \;\geq\;& \tfrac{1}{2}\sum_{i=1}^{K-1}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_{K-1}^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_{K-1}^{-1}\right\}.
\end{aligned} \tag{3.5}
$$

From the recursive formula (3.1) we obtain:

(3.6)

$$
\begin{aligned}
\mathcal{T}_K \;\geq\;& \tfrac{1}{2}\sum_{i=1}^{K-1}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_{K-1}^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_{K-1}^{-1}\right\}+ \\
& \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}+\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\} \\
\geq\;& \tfrac{1}{2}\sum_{i=1}^{K-1}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}+\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\} \\
=\;& \tfrac{1}{2}\sum_{i=1}^{K-1}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}+\tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\} \\
=\;& \tfrac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}+\tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+\\
&\quad \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}+\mathbf{A}_K-\mathbf{Y}_{K-1}-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}+\mathbf{A}_K-\mathbf{Y}_{K-1}-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}\\
&= \tfrac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+\frac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{X}_K^{-1}\right\}
\end{aligned}
$$

The second inequality follows from applying Lemma 2.1 for the second term on the right and considering that $\mathbf{X}_K = \mathbf{X}_{K-1} + \mathbf{A}_K$, $\mathbf{Y}_K = \mathbf{Y}_{K-1} + \mathbf{B}_K$. The third equality follows from Lemma 2.2. Thus, we have proven the desired result.

The second step of the proof is straightforward. From (3.2), it is easy to check that $\mathcal{T}_K \geq 0$ (all the terms of the form $\mathrm{Tr}\left\{\mathbf{X}\mathbf{B}^{-1}\mathbf{X}\mathbf{A}^{-1}\right\}$ with $\mathbf{X} = \mathbf{X}^H$, $\mathbf{A} \succ \mathbf{0}$, $\mathbf{B} \succ \mathbf{0}$ can be re-written as $\mathrm{Tr}(\mathbf{N}\mathbf{N}^H) \geq 0$ with $\mathbf{N} = \mathbf{A}^{-1/2}\mathbf{X}\mathbf{B}^{-1/2}$).

References

[1] E. V. BELMEGA, S. LASAULCE, and M. DEBBAH, "A Trace Inequality for Positive Definite Matrices", *Journal of inequalities in pure and applied mathematics (JIPAM)*, Vol. 10, No. 1, Article 5, 4 pp., (2009).

[2] E. V. BELMEGA, S. LASAULCE, M. DEBBAH, M. JUNGERS, and J. DUMONT, "Power allocation games in wireless networks of multi-antenna terminals", *Springer Telecommunications Systems Journal, VALUETOOLS special issue*, revised (minor revisions), (2009).

[3] J. ROSEN, "Existence and uniqueness of equilibrium points for concave n-person games", *Econometrica*, Vol. 33, pp. 520–534, (1965).

[4] K. M. ABADIR AND J. R. MAGNUS, "Matrix Algebra", *Cambridge University Press, New York, USA*, pp. 329, pp. 338, (2005).

Université Paris-Sud XI, SUPELEC, Laboratoire des Signaux et Systèmes, Gif-sur-Yvette, France.
E-mail address: belmega@lss.supelec.fr
URL: http://veronica.belmega.lss.supelec.fr

CNRS, ENSEM, CRAN, Vandoeuvre, France.
E-mail address: marc.jungers@cran.uhp-nancy.fr
URL: http://perso.ensem.inpl-nancy.fr/Marc.Jungers/

CNRS, SUPELEC, Laboratoire des Signaux et Systèmes, Gif-sur-Yvette, France.
E-mail address: lasaulce@lss.supelec.fr
URL: http://samson.lasaulce.lss.supelec.fr